20071116 Summary

① CS	② open GW	③ closed GW	④ U(1)-instanton counting on $\mathbb{R}^4$
S^3	$T^*S^3 \supset S^3$	$\mathcal{O}(-1)\oplus\mathcal{O}(-1)$	
N, k	g_s, t	g_s, Q	g_s, Q

$$\frac{2\pi i}{k+N} = g_s \ , \ \frac{2\pi i N}{k+N} = t \ , \ Q = e^{-t}$$

$$\left(q \text{ (quantum group param.)} = e^{\frac{g_s}{2}} \right)$$

① SU(N) CS partition function of M^3, level k

$$Z_{SU(N),k}(M^3) = \int_{\mathcal{A}/\mathcal{G}} e^{2\pi i k CS(A)} DA$$

①' exact solution ← Both are defined rigorously.

①'' perturbation theory ←

$$Z_{SU(N),k}(M) \underset{k\to\infty}{\sim} \begin{pmatrix} \text{stationary} \\ \text{phase} \end{pmatrix} \times \exp\left[\sum_{r=1}^{\infty} S_r \left(\frac{2\pi i}{k+N}\right)^r \right]$$
or 1-loop

② open GW invariants T^*M^3 with lagrangian M^3

$$\log Z(S^3) = \sum_{\substack{h = \# \text{ of } \partial\Sigma \\ g = \text{genus}}} t^h g_s^{2g-2} \#\{f : (\Sigma, \partial\Sigma) \to (T^*S^3, S^3)\}$$

So far, not defined rigorously

③ closed GW invariants of $\mathcal{O}(-1)\oplus\mathcal{O}(-1) \to \mathbb{P}^1$

$$\log Z(\text{resolv. conifold}) = \sum_{\substack{g, d \\ d \geq 1}} g_s^{2g-2} Q^d \#\{f : \Sigma_g \to \text{resolv. conifold}\}$$

defined rigorously

④ $U(1)$-instanton counting (K-theory)

$\mathbb{C}^2 \curvearrowleft \mathbb{C}^*$ $\quad (e^{ig_s}x, e^{-ig_s}y)$

$$Z = \sum_{k\geq 0} \mathrm{ch}\, \mathbb{C}[\overline{M}(k, U(1))]\, Q^k$$

↑ defined rigorously

$\boxed{\text{Dualities : ①', ①'', ②, ③, ④ are all equal. } (M=S^3) \text{ (except possibly ②)}}$

In the simplest case (i.e. $M=S^3$ without link)

we can explicitly compute

①' $\quad Z_{SU(N),k}(S^3) = S_{00}$ (S-matrix in CFT)

③, ④ $\quad \exp\left(-\sum_{n=1}^{\infty} \frac{Q^n}{n(e^{\frac{ng_s}{2}} - e^{-\frac{ng_s}{2}})^2}\right)$

We can check they are equal (modulo perturb. terms) by direct computation.

Sometimes we need an <u>analytic continuation</u>.

CS : N : fix $\quad k$: large

g_s : small , $t = Ng_s$: small

closed GW instanton $\quad g_s$: small , $Q = e^{-t}$: small

Identification of invariants by physical intuitions:

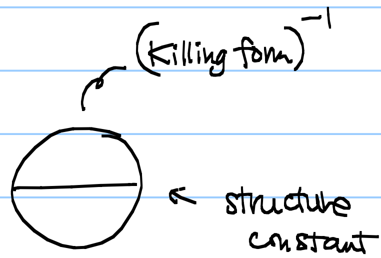
① $'$ = ① $''$: perturbation expansion
not rigorous so far, but if we can expand the path integral as in the fin. dim. integral, it is OK.

① $''$ = ② (Witten) string field theory
(This is rather difficult.) even for physicists.

$\Rightarrow$ mathematical framework by Fukaya

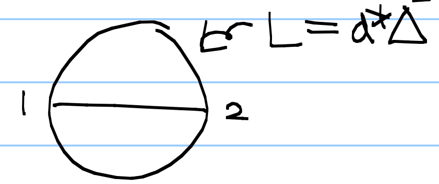
$$S_r = \sum_{\substack{\Gamma : \text{trivalent graph} \\ \text{connected with } 2r \text{ vertex}}} a_\Gamma(SU(N)) \cdot I_\Gamma(M)$$

a_Γ :



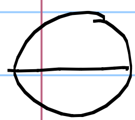
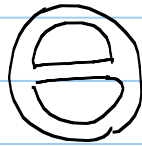
$f_{stu} f_{xyz} g^{sx} g^{ty} g^{uz}$

I_Γ



$$\int_{M\times M} L(x_1,x_2) L(x_1,x_2) L(x_1,x_2)$$

idea : we are at "degenerate" situations
Let's perturb propagator/lagrangian
by Morse functions!

 v.s. 

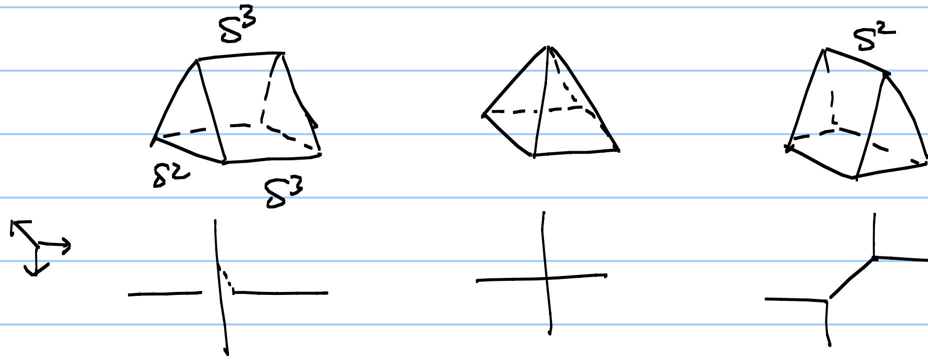
- propagator : $\Delta^{-1} \rightsquigarrow e^{\frac{1}{\varepsilon}f}\Delta^{-1}e^{-\frac{1}{\varepsilon}f}$
  (cf. Witten : Morse theory)
- lagrangian $M_{\subset T^*M} \rightsquigarrow$ graph of εdf

Since these are topological theories, invariants are independent of ε.

CS : $\varepsilon \to \infty$ original (Morse flow)
$\varepsilon \to 0$ counting of gradient graphs
lagrangian : $\varepsilon \to 0$ Riemann surfaces become thin.
$\longrightarrow$ gradient graphs

② = ③ • 'tHooft "filling hole"
$\underbrace{t^{h}}_{\text{fix \& sum up over } h} g_s^{2g-2}$

- geometric transition
$T^*S^3 \underset{\text{deform}}{\rightsquigarrow} xy = zw \underset{\text{resolu.}}{\longleftarrow} \mathcal{O}(-1)\oplus\mathcal{O}(-1)$
$\downarrow$
$\mathbb{P}^1$



This geometric explanation is very appealing, but note that we need to make change of variables as

$$e^{-t} = Q$$

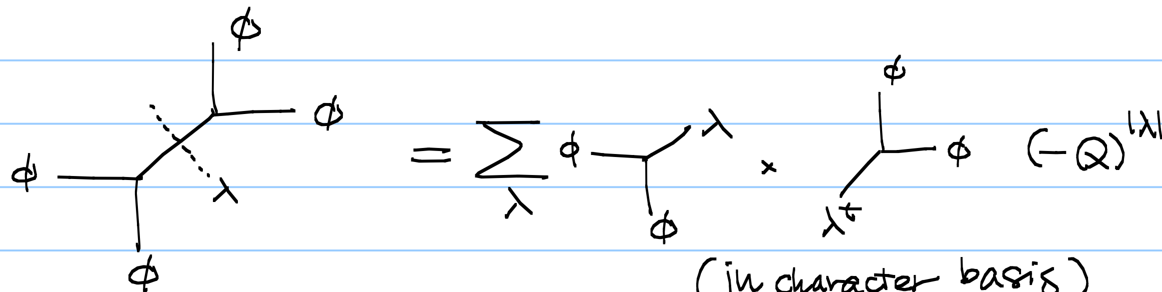
t: small in open GW $\qquad$ Q: small in closed GW.

Andrew's comment: Ooguri-Vafa hep-th/0205297

③ = ④ geometric engineering

In this case, both invariants are exactly equal.
Moreover combinatorial expressions are the same.
Thus geometric picture is not clear in this duality,
but computational relation is VERY strong.

- GW side



(in character basis)

- instanton counting

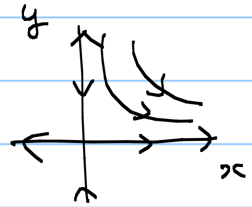
localization formula

$$\mathrm{Hilb}^n\mathbb{C}^2 \longrightarrow \underset{\text{resolution}}{S^n\mathbb{C}^2} \qquad \mathcal{M}(n,\mathcal{O}(1)) = S^n\mathbb{C}^2$$

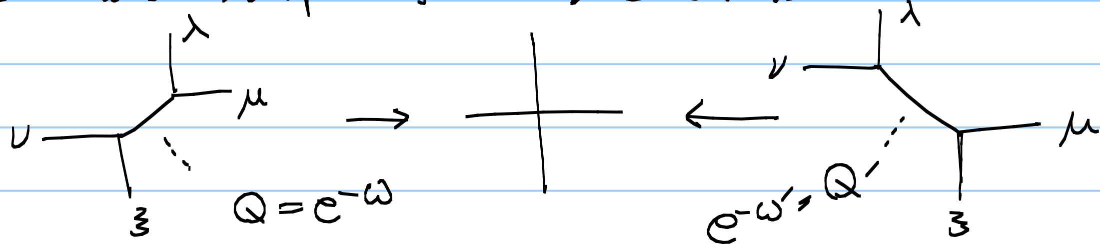
$$\mathrm{ch}\,\mathbb{C}[S^n\mathbb{C}^2] = \mathrm{ch}\,(-1)^* H^*(\mathrm{Hilb}^n\mathbb{C}^2)$$

$$= \sum_{\lambda} \frac{1}{\mathrm{ch}\,\Lambda_{-1} T^*_{I_\lambda}\mathrm{Hilb}^n\mathbb{C}^2}$$

$$T_{I_\lambda}\mathrm{Hilb}^n\mathbb{C}^2 = \sum_{s\in\lambda} \underbrace{e^{g_s h(s)}}_{\text{stable mfd}} + \underbrace{e^{-g_s h(s)}}_{\text{unstable mfd}}$$



° One more example of analytic continuation



$$\frac{Z_{\lambda\mu\nu\xi}(g_s, Q)}{Z_{\phi\phi\phi\phi}(g_s, Q)} \underset{\text{simple factors}}{\sim} \frac{Z_{\lambda\mu\nu\xi}(g_s, Q')}{Z_{\phi\phi\phi\phi}(g_s, Q')} \quad \text{if } Q' = Q^{-1}$$

Remark The "Chain" of dualities still continue .....

- ③ : A-model $\overset{\exists}{\underset{\text{mirror}}{\longleftrightarrow}}$ B-model

  We discuss genus 0 part in Dec.
  $\rightsquigarrow$ SW curve.

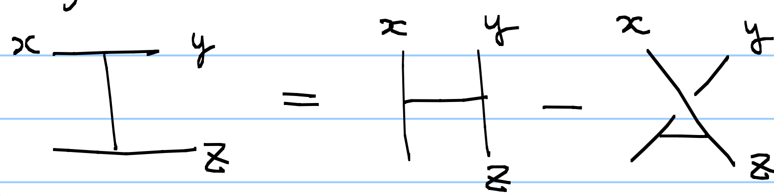
- GW v.s. Gopakumar-Vafa invariants (BPS counting)
  or Donaldson-Thomas invariants

- CS perturbation theory
  $\longrightarrow$ { Rozansky-Witten invariants, finite type invariants

  For the proof of the topological invariance of

  $$S_r = \sum a_p(\mathfrak{g}) I_p(M^3),$$

we only need the IHX relation on $a_\Gamma(\mathfrak{g})$

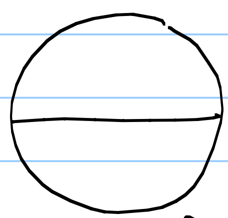


$$[[x,y],z] = [x,[y,z]] - [y,[x,z]]$$

(Jacobi identity)

If we have an assignment $\{\Gamma \mapsto a_\Gamma\}$ satisfying this relation, we get a topological invariant.

Such an assign. was given by Rozansky–Witten for a given hyperKähler mfd (X,ω).



← put a curvature $R^{j}_{ik\bar{\ell}}$

(holo. symplect form)$^{-1}$

$\rightsquigarrow$ get $\tilde{a}_\Gamma(X) \in \Omega^{0,2r}(X)$

$$\int_X \omega^r \tilde{\Gamma}(X) =: a_\Gamma(X)$$

IHX rel. follows from the Bianchi identity.

universal invariant

$$\left(\text{trivalent graph} / \text{IHX}\right)^* \ni \{\Gamma \to a_\Gamma \in \mathbb{R}\}$$

I said invariants are all equal. But this statement is not quite true.

Disagreements comes from the constant map contributions to the GW invariants

$$\mathcal{M}_{g,0}(X) \cong \underset{\substack{\uparrow \\ \text{noncpt}}}{X} \times \mathcal{M}_{g,0} \qquad [\mathcal{M}_{g,0}(X)]^{vir} = e(T_X)[X] \int_{\mathcal{M}_{g,0}} c_{g-1}(\mathbb{E})^3$$

If X would be compact, $e(T_X)[X] = e(X)$.

In our case we should formally define the constant map contribution by the same formula.

There is still mismatch in genus $0, 1$
(In these cases $\mathcal{M}_{g,0} = \phi$)

But certainly there should be something, if the invariants are defined via the path integral.
It should be possible to compute them via the perturvative expansion ?

Generalization / Variants

• framing

– choice of trivialization of the tangent b'dle of M^3

② ?

③ : ---- normal b'dle of $\mathbb{P}^1$ $\qquad \mathcal{O}(-1)\oplus\mathcal{O}(-1)$: std

$\qquad\qquad \rightsquigarrow \mathcal{O}(n)\oplus\mathcal{O}(-n-2)$

④ : --- twist by the line b'dle $\mathcal{L} = \det \mathcal{O}^{[k]}$, where $\mathcal{O}^{[k]}$: v.b/Hilb^k $\quad$ fiber at $I = \mathbb{C}[x,y]/I$

• link

① SU(N) CS partition function of (S^3, L), $L = L_1 \cup \dots \cup L_\ell$

level $k \in \mathbb{Z}$

$$Z_{k;R_1,\dots,R_\ell}(S^3,L) = \int_{\mathcal{A}/\mathcal{G}} e^{2\pi i k\, CS(A)}\, \mathrm{tr}_{R_i} \mathrm{Hol}_{L_i}(A)\, DA$$

①' exact solution

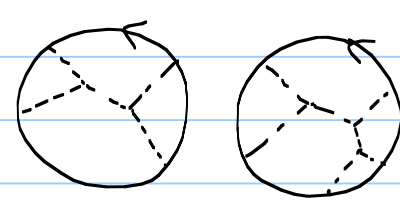
Hilbert space = conformal blocks

e.g. $Z_k(S^3) = S_{00}$, $Z_{k;R}(\text{unknot}) = S_{0R}$,

$Z_{k;R_iR_j}($  $) = S_{R_iR_j}$

①'' perturbation theory

$Z_{k;\vec{R}}(S^3,L) \underset{k\to\infty}{\sim}$



cord diagram

② C_{L_i} : conormal bdle to $L_i \subset T^*S^3$

open GW inv. of T^*S^3 with $\bigcup_i C_{L_i}$

③ $\tilde{C}_{L_i}$: transition of C_{L_i}

④? Sergei : $U(1)$ instanton with singularities along axis

N : vector bundles over Hilb^n

(or cpx)

• $\Gamma \subset SU(2)$

① : CS partition functions on S^3/Γ

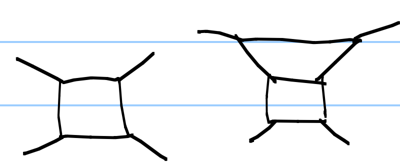
(NB. flat connection $\sim \bigoplus_i \rho_i^{\oplus N_i}$ ρ_i : irr. rep.

② ?

③ $\overline{(\mathcal{O}(-1)\oplus\mathcal{O}(-1))}/\Gamma$

$\downarrow$

$\mathbb{P}^1$



(how about different model ?)

④ gauge theory partition function for $G = G(\Gamma)$ via McKay

$N_i \leftrightarrow a_i$